SM311O – First Exam

Feb 19, 1997

1. (**15 points**) Let A be a 2×2 real-valued constant matrix, one of whose eigenvalues is $1 - 2i$ with the corresponding eigenvector
$$\begin{bmatrix} 1 \\ 1-i \end{bmatrix}.$$
Write down the general solution to the system of differential equations $\mathbf{x}' = A\mathbf{x}$.

2. (**15 points**) Let f be defined by
$$f(x) = \begin{cases} -2 & \text{if } 0 \le x < 3 \\ 1 & \text{if } 3 \le x < 4. \end{cases}$$
Determine the first two nonzero terms of the Fourier Sine series of f.

3. (**15 points**) Let $f(x,y) = -x^2y^3 + 2x^3y^2$.
   (a) Find the gradient of f at $P = (1,2)$.
   (b) Find the directional derivative of f at $P = (1,2)$ in the direction of $\mathbf{e} = \langle 1, -1 \rangle$.
   (c) Find the direction of maximum ascent at $P = (-1,3)$.

4. (**10 points**) Let S be a sphere of radius 3. Find a normal vector to S at 60 degrees latitude and 30 degrees longitude.

5. (**15 points**) Let $\mathbf{v} = \langle \frac{ay}{\sqrt{x^2+y^2}}, \frac{bx}{\sqrt{x^2+y^2}} \rangle$ be the velocity field of a fluid, where a and b are constants. Find all nonzero values of these constants for which $\mathbf{v}$ is incompressible.

6. (**30 points**) Let $\mathbf{v} = \langle 3y^2 - x^2, x^2 + 2xy \rangle$.
   (a) Does $\mathbf{v}$ have a stream function? If no, explain why not. If yes, determine it.
   (b) Consider the fluid particle that is located at $(1,-1)$ at time 0. Write down the *Mathematica* commands you would use to plot the graph of the path of this particle for $0 < t < 3$.

Bonus Points

Let S be a sphere of radius 6000 kilometers.

1. (**10 points**) State a parametrization of a typical circle that passes through both poles.

2. (**10 points**) State a parametrization of the circle located on the sphere and in the plane $z = 3000$.